

Global well-posedness for the incompressible viscoelastic fluids in the critical L^p framework

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Abstract

We investigate global strong solutions for the incompressible viscoelastic system of Oldroyd–B type with the initial data close to a stable equilibrium. We obtain the existence and uniqueness of the global solution in a functional setting invariant by the scaling of the associated equations, where the initial velocity has the same critical regularity index as for the incompressible Navier–Stokes equations, and one more derivative is needed for the deformation tensor. Like the classical incompressible Navier–Stokes, one may construct the unique global solution for a class of large highly oscillating initial velocity. Our result also implies that the deformation tensor F has the same regularity as the density of the compressible Navier–Stokes equations.

1 Introduction

In this paper, we consider the following system describing incompressible viscoelastic fluids.

$$\begin{cases} \nabla \cdot v = 0, & x \in \mathbb{R}^N, \ N \geq 2, \\ v_t + v \cdot \nabla v + \nabla p = \mu \Delta v + \nabla \cdot \left[\frac{\partial W(F)}{\partial F} F^\top \right], \\ F_t + v \cdot \nabla F = \nabla v F, \\ F(0, x) = I + E_0(x), \ v(0, x) = v_0(x). \end{cases} \quad (1.1)$$

Here, v , p , $\mu > 0$, F and $W(F)$ denote, respectively, the velocity field of materials, pressure, viscosity, deformation tensor and elastic energy functional. The third equation is simply the consequence of the chain law. It can also be regarded as the consistence condition of the flow trajectories obtained from the velocity field v and also of those obtained from the deformation tensor F [11, 17, 22, 27, 29]. Moreover, on the right-hand side of the momentum equation, $\frac{\partial W(F)}{\partial F}$ is the Piola–Kirchhoff stress tensor and $\frac{\partial W(F)}{\partial F} F^\top$ is the Cauchy–Green tensor. The latter is the change variable (from Lagrangian to Eulerian coordinates) form of the former one [22]. The above system is equivalent to the usual Oldroyd–B model for viscoelastic fluids in the case of infinite Weissenberg number [21]. On the other hand, without the viscosity term, it represents exactly the incompressible elasticity in Eulerian coordinates. We refer to [3, 11, 21, 26, 28, 29] and their references for the detailed derivation and physical background of the above system.

Throughout this paper, we will use the notations of

$$(\nabla v)_{ij} = \frac{\partial v_i}{\partial x_j}, \ (\nabla v F)_{ij} = (\nabla v)_{ik} F_{kj}, \ (\nabla \cdot F)_i = \partial_j F_{ij},$$

and summation over repeated indices will always be understood.

For incompressible viscoelastic fluids, Lin et al. [27] proved the global existence of classical small solutions for the two-dimensional case with the initial data $v_0, E_0 = F_0 - I \in H^k(\mathbb{R}^2)$, $k \geq 2$, by introducing an auxiliary vector field to replace the transport variable F . Using the method in [19] for the damped wave equation, they [27] also obtained the global existence of classical small solutions for the three-dimensional case with the initial data $v_0, E_0 \in H^k(\mathbb{R}^3)$, $k \geq 3$. Lei and Zhou [24] obtained the same

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results for the two-dimensional case with the initial data $v_0, E_0 \in H^s(\mathbb{R}^2)$, $s \geq 4$, via the incompressible limit working directly on the deformation tensor F . Then, Lei et al. [23] proved the global existence for two-dimensional small-strain viscoelasticity with $H^2(\mathbb{R}^2)$ initial data and without assumptions on the smallness of the rotational part of the initial deformation tensor. It is worth noticing that the global existence and uniqueness for the large solution of the two-dimensional problem is still open. Recently, by introducing an auxiliary function $w = \Delta v + \frac{1}{\mu} \nabla \cdot E$, Lei et al. [22] obtained a weak dissipation on the deformation F and the global existence of classical small solutions to N -dimensional system with the initial data $v_0, E_0 \in H^2(\mathbb{R}^N)$ and $N = 2, 3$. All these results need that the initial data v_0 and E_0 have the same regularity, and the regularity index of the initial velocity v_0 is bigger than the critical regularity index for the classical incompressible Navier–Stokes equations.

There are some classical results for the following incompressible Navier–Stokes equations.

$$\begin{cases} v_t + v \cdot \nabla v + \nabla p = \mu \Delta v, \\ \nabla \cdot v = 0, \\ v(0, x) = v_0(x). \end{cases} \quad (1.2)$$

In 1934, J. Leray proved the existence of global weak solutions for (1.2) with divergence-free $u_0 \in L^2$ (see [25]). Then, H. Fujita and T. Kato (see [16]) obtain the uniqueness with $u_0 \in \dot{H}^{\frac{N}{2}-1}$. The index $s = N/2 - 1$ is critical for (1.2) with initial data in \dot{H}^s : this is the lowest index for which uniqueness has been proved (in the framework of Sobolev spaces). This fact is closely linked to the concept of scaling invariant space. Let us precise what we mean. For all $l > 0$, system (1.2) is obviously invariant by the transformation

$$u(t, x) \rightarrow u_l(t, x) := lu(l^2t, lx), u_0(x) \rightarrow u_{0l}(x) := lu_0(lx),$$

and a straightforward computation shows that $\|u_0\|_{\dot{H}^{\frac{N}{2}-1}} = \|u_{0l}\|_{\dot{H}^{\frac{N}{2}-1}}$. This idea of using a functional setting invariant by the scaling of (1.2) is now classical and originated many works. In [5], M. Cannone, Y. Meyer and F. Planchon proved that: if the initial data satisfy

$$\|v_0\|_{\dot{B}_{p,\infty}^{-1+\frac{N}{p}}} \leq c\nu,$$

for $p > N$ and some constant c small enough, then the classical Navier–Stokes system (NS) is globally wellposed. Refer to [4, 9, 10, 20] for a recent panorama.

In this paper, using the method in [6, 7] studying the compressible Navier–Stokes system, we are concerned with the existence and uniqueness of a solution for the initial data in a functional space with minimal regularity order $v_0 \in \dot{B}_{p,1}^{-1+\frac{N}{p}}$. Although the equation (1.1)₃ for the deformation tensor F is identical to the equation for the vorticity $\omega = \nabla \times v$ of the Euler equations

$$\partial_t \omega + v \cdot \nabla \omega = \nabla v \omega,$$

our result implies that the deformation tensor F has the same regularity as the density of the compressible Navier–Stokes equations.

At this stage, we will use scaling considerations for (1.1) to guess which spaces may be critical. We observe that (1.1) is invariant by the transformation

$$(v_0(x), F_0(x)) \rightarrow (lv_0(lx), F_0(lx)),$$

$$(v(t, x), F(t, x), P(t, x)) \rightarrow (lv(l^2t, lx), F(l^2t, lx), l^2P(l^2t, lx)),$$

up to a change of the elastic energy functional W into l^2W .

Definition 1.1. A functional space $E \subset (\mathcal{S}'(\mathbb{R}^N))^N \times (\mathcal{S}'(\mathbb{R}^N))^{N \times N}$ is called a critical space if the associated norm is invariant under the transformation $(v(x), F(x)) \rightarrow (lv(lx), F(lx))$ (up to a constant independent of l).

In this paper, $(\dot{B}_{p,1}^{\frac{N}{p}-1})^N \times (\dot{B}_{p,1}^{\frac{N}{p}})^{N \times N}$, $p \in [2, \infty)$, is a critical space.

Assume that $E_0 := F_0 - I$ and v_0 satisfy the following constraints:

$$\nabla \cdot v_0 = 0, \det(I + E_0) = 1, \nabla \cdot E_0^\top = 0, \quad (1.3)$$

and

$$\partial_m E_{0ij} - \partial_j E_{0im} = E_{0lj} \partial_l E_{0im} - E_{0lm} \partial_l E_{0ij}. \quad (1.4)$$

The first three of these expressions are just the consequences of the incompressibility condition and the last one can be understood as the consistency condition for changing variables between the Lagrangian and Eulerian coordinates [22].

For simplicity, we only consider the case of Hookean elastic materials: $W(F) = |F^2|$. Define the usual strain tensor by the form

$$E = F - I. \quad (1.5)$$

Then, the system (1.1) is

$$\begin{cases} \nabla \cdot v = 0, & x \in \mathbb{R}^N, \quad N \geq 2, \\ v_{it} + v \cdot \nabla v_i + \partial_i p = \mu \Delta v_i + E_{jk} \partial_j E_{ik} + \partial_j E_{ij}, \\ E_t + v \cdot \nabla E = \nabla v E + \nabla v, \\ (v, E)(0, x) = (v_0, E_0)(x). \end{cases} \quad (1.6)$$

The global well-posedness result had been obtained with the small initial data $v_0 \in (\dot{B}_{2,1}^{\frac{N}{2}-1})^N$ and $E_0 \in (\dot{B}_{2,1}^{\frac{N}{2}} \cap \dot{B}_{2,1}^{\frac{N}{2}-1})^{N \times N}$ independently in [30, 35].

Theorem 1.1 ([30, 35]). *Suppose that the initial data satisfies the incompressible constraints (1.3), $v_0 \in (\dot{B}_{2,1}^{\frac{N}{2}-1})^N$ and $E_0 \in (\dot{B}_{2,1}^{\frac{N}{2}})^{N \times N}$. Then there exist $T > 0$ and a unique local solution for system (1.6) that satisfies*

$$(v, E) \in U_T^{\frac{N}{2}}, \quad \|(v, E)\|_{U_T^{\frac{N}{2}}} \leq C(\|E_0\|_{\dot{B}_{2,1}^{\frac{N}{2}}} + \|v_0\|_{\dot{B}_{2,1}^{\frac{N}{2}-1}}), \quad (1.7)$$

and

$$\nabla \cdot v = 0, \quad \det(I + E) = 1, \quad \nabla \cdot E^\top = 0, \quad (1.8)$$

where

$$U_T^s = \left(L^1([0, T]; \dot{B}_{2,1}^{s+1}) \cap C([0, T]; \dot{B}_{2,1}^{s-1}) \right)^N \times \left(C([0, T]; \dot{B}_{2,1}^s) \right)^{N \times N},$$

Furthermore, suppose that the initial data satisfies the incompressible constraints (1.3)–(1.4), $v_0 \in (\dot{B}_{2,1}^{\frac{N}{2}-1})^N$, $E_0 \in (\dot{B}_{2,1}^{\frac{N}{2}} \cap \dot{B}_{2,1}^{\frac{N}{2}-1})^{N \times N}$ and

$$\|E_0\|_{\dot{B}_{2,1}^{\frac{N}{2}} \cap \dot{B}_{2,1}^{\frac{N}{2}-1}} + \|v_0\|_{\dot{B}_{2,1}^{\frac{N}{2}-1}} \leq \lambda, \quad (1.9)$$

where λ is a small positive constant. Then there exists a unique global solution for system (1.6) that satisfies

$$(v, E) \in V^{\frac{N}{2}}, \quad \|(v, E)\|_{V^{\frac{N}{2}}} \leq C(\|E_0\|_{\dot{B}_{2,1}^{\frac{N}{2}} \cap \dot{B}_{2,1}^{\frac{N}{2}-1}} + \|v_0\|_{\dot{B}_{2,1}^{\frac{N}{2}-1}}), \quad (1.10)$$

where

$$V^s = \left(L^1(\mathbb{R}^+; \dot{B}_{2,1}^{s+1}) \cap C(\mathbb{R}^+; \dot{B}_{2,1}^{s-1}) \right)^N \times \left(L^2(\mathbb{R}^+; \dot{B}_{2,1}^s) \cap C(\mathbb{R}^+; \dot{B}_{2,1}^s \cap \dot{B}_{2,1}^{s-1}) \right)^{N \times N}.$$

In this paper, we will obtain the following theorem, where

$$f^l = \sum_{2^k \leq R_0} \Delta_k f, \quad f^h = \sum_{2^k > R_0} \Delta_k f,$$

and R_0 is as in Proposition 4.2.

Theorem 1.2. Suppose that the initial data satisfies the incompressible constraints (1.3)–(1.4), $E_0 \in \dot{B}_{p_1,1}^{\frac{N}{p_1}}$, $E_0^l \in \dot{B}_{2,r}^s$, $E_0^h \in \dot{B}_{2,r}^{s+1}$, $v_0 \in \dot{B}_{p_1,1}^{\frac{N}{p_1}-1} \cap \dot{B}_{2,r}^s$, for some $p_1 \in [2, \infty)$, $p_2 \in [p_1, \infty)$, $r \in [1, \infty]$ and $s \in \mathbb{R}$ such that

$$s \in \begin{cases} (-1, \frac{N}{2} - 1) & \text{if } r > 1, \\ (-1, \frac{N}{2} - 1] & \text{if } r = 1. \end{cases} \quad (1.11)$$

There exists a constant λ depending only on N , p_2 and s such that if

$$\|E_0^l\|_{\dot{B}_{2,r}^s} + \|E_0^h\|_{\dot{B}_{2,r}^{s+1} \cap \dot{B}_{p_2,1}^{\frac{N}{p_2}-1}} + \|v_0\|_{\dot{B}_{2,r}^s} + \|v_0^h\|_{\dot{B}_{p_2,1}^{\frac{N}{p_2}-1}} \leq \lambda, \quad (1.12)$$

then there exists a global solution for system (1.6) that satisfies

$$(v, E) \in V_{p_1,r}^s$$

where

$$\begin{aligned} V_{p,r}^s &= \left\{ v \in \left(\tilde{C}(\mathbb{R}^+; \dot{B}_{2,r}^s \cap \dot{B}_{p,1}^{\frac{N}{p}-1}) \cap \tilde{L}^1(\mathbb{R}^+; \dot{B}_{2,r}^{s+2} \cap \dot{B}_{p,1}^{\frac{N}{p}+1}) \right)^N \right. \\ &\quad \left. E \in \left(\tilde{C}(\mathbb{R}^+; \dot{B}_{2,r}^{s+1} \cap \dot{B}_{p,1}^{\frac{N}{p}}) \cap L^2(\mathbb{R}^+; \dot{B}_{p,1}^{\frac{N}{p}}) \right)^{N \times N}, \right. \\ &\quad \left. E^l \in \left(\tilde{L}^2(\mathbb{R}^+; \dot{B}_{2,r}^{s+1}) \cap \tilde{L}^1(\mathbb{R}^+; \dot{B}_{2,r}^{s+2}) \right)^{N \times N}, E^h \in \left(\tilde{L}^1(\mathbb{R}^+; \dot{B}_{2,r}^{s+1}) \right)^{N \times N} \right\}. \end{aligned}$$

Furthermore, the solution is unique in $V_{p_1,r}^s$ if $p_1 \leq 2N$.

Remark 1.1. This result allows us to construct the unique global solution for the high oscillating initial velocity v_0 . For example,

$$v_0(x) = \varepsilon^{\frac{N}{p}-1} \sin\left(\frac{x_1}{\varepsilon}\right) \phi(x), \quad \phi(x) \in \mathcal{S}(\mathbb{R}^N), \quad p \in [N, \infty),$$

which satisfies

$$\|v_0\|_{\dot{B}_{2,r}^s} + \|v_0^h\|_{\dot{B}_{p_2,1}^{\frac{N}{p_2}-1}} \ll 1,$$

if $s < -1 + \frac{N}{p}$, $p_2 > p$, ε is small enough, (Proposition 2.9, [7]).

Remark 1.2. The L^2 -decay in time for E is a key point in the proof of the global existence. We shall also get a L^1 -decay in a space $E^l \in \left(\tilde{L}^1(\mathbb{R}^+; \dot{B}_{2,r}^{s+2}) \right)^{N \times N}$, $E^h \in \left(\tilde{L}^1(\mathbb{R}^+; \dot{B}_{2,r}^{s+1}) \right)^{N \times N}$.

Remark 1.3. Theorem 1.2 implies that the deformation tensor F has similar property as the density of the compressible Navier–Stokes system [6, 7]. And we think that the incompressible viscoelastic system is similar to the compressible Navier–Stokes system.

Remark 1.4. Similar to the compressible Navier–Stokes system [6, 7], the initial data do not really belong to a critical space in the sense of Definition 1.1. We indeed made the additional assumptions $E_0^l \in \dot{B}_{2,r}^s$, $E_0^h \in \dot{B}_{2,r}^{s+1}$, $v_0 \in \dot{B}_{2,r}^s$. On the other hand, our scaling considerations do not take care of the Cauchy–Green tensor term. A careful study of the linearized system (see Proposition 4.2 below) besides indicates that such an assumption may be unavoidable.

Remark 1.5. Considering the general viscoelastic model (1.1), if the strain energy function satisfies the strong Legendre–Hadamard ellipticity condition

$$\frac{\partial^2 W(I)}{\partial F_{il} \partial F_{jm}} = (\alpha^2 - 2\beta^2) \delta_{il} \delta_{jm} + \beta^2 (\delta_{im} \delta_{jl} + \delta_{ij} \delta_{lm}), \quad \text{with } \alpha > \beta > 0, \quad (1.13)$$

and the reference configuration stress-free condition

$$\frac{\partial W(I)}{\partial F} = 0, \quad (1.14)$$

then we can obtain the same results as that in Theorem 1.2.

In this paper, we introduce the following function:

$$c = \Lambda^{-1} \nabla \cdot E,$$

where $\Lambda^s f = \mathcal{F}^{-1}(|\xi|^s \hat{f})$. Then, the system (1.6) reads

$$\begin{cases} \nabla \cdot v = \nabla \cdot c = 0, & x \in \mathbb{R}^N, \ N \geq 2, \\ v_{it} + v \cdot \nabla v_i + \partial_i p = \mu \Delta v_i + E_{jk} \partial_j E_{ik} + \Lambda c_i, \\ c_t + v \cdot \nabla c + [\Lambda^{-1} \nabla \cdot, v] \nabla E = \Lambda^{-1} \nabla \cdot (\nabla v E) - \Lambda v, \\ \Delta E_{ij} = \Lambda \partial_j c_i + \partial_k (\partial_k E_{ij} - \partial_j E_{ik}), \\ (v, c)(0, x) = (v_0, \Lambda^{-1} \nabla \cdot E_0)(x). \end{cases} \quad (1.15)$$

So, we need to study the following mixed parabolic–hyperbolic linear system with a convection term:

$$\begin{cases} \nabla \cdot v = \nabla \cdot c = 0, & x \in \mathbb{R}^N, \ N \geq 2, \\ v_{it} + u \cdot \nabla v_i + \partial_i p - \mu \Delta v_i - \Lambda c_i = G, \\ c_t + u \cdot \nabla c + \Lambda v = L, \\ (v, c)(0, x) = (v_0, c_0)(x), \end{cases} \quad (1.16)$$

where $\operatorname{div} u = 0$ and $c_0 = \Lambda^{-1} \nabla \cdot E_0$. This system is similar to the system

$$\begin{cases} c_t + v \cdot \nabla c + \Lambda d = F, \\ d_t + v \cdot \nabla d - \bar{\mu} \Delta d - \Lambda c = G, \end{cases} \quad (1.17)$$

for compressible Navier–Stokes system [6, 7]. Using the similar method studying the compressible Navier–Stokes system in [6, 7], we can obtain some important estimates for the system (1.16).

As for the related studies on the existence of solutions to nonlinear elastic systems, there are works by Sideris [31] and Agemi [1] on the global existence of classical small solutions to three–dimensional compressible elasticity, under the assumption that the nonlinear terms satisfy the null conditions. The global existence for three–dimensional incompressible elasticity was then proved via the incompressible limit method in [32] and by a different method in [33]. It is worth noticing that the global existence and uniqueness for the corresponding two–dimensional problem is still open.

As for the density-dependent incompressible viscoelastic fluids, Hu and Wang [18] obtained the existence and uniqueness of the global strong solution with small initial data in \mathbb{R}^3 .

The rest of this paper is organized as follows. In Section 2, we state three lemmas describing the intrinsic properties of viscoelastic system. In Section 3, we present the functional tool box: Littlewood–Paley decomposition, product laws in Sobolev and hybrid Besov spaces. The next section is devoted to the study of some linear models associated to (1.6). In Section 5, we give some a priori estimates. At last, we will study the global well-posedness for (1.6).

2 Basic mechanics of viscoelasticity

Using the similar arguments as that in the proof of Lemmas 1–3 in [22], we can easily obtain the following three lemmas.

Lemma 2.1. *Assume that $\det(I + E_0) = 1$ is satisfied and (v, F) is the solution of system (1.6). Then the following is always true:*

$$\det(I + E) = 1, \quad (2.1)$$

for all time $t \geq 0$, where the usual strain tensor $E = F - I$.

Lemma 2.2. *Assume that $\nabla \cdot E_0^\top = 0$ is satisfied, then the solution (v, F) of system (1.6) satisfies the following identities:*

$$\nabla \cdot F^\top = 0, \quad \text{and} \quad \nabla \cdot E^\top = 0, \quad (2.2)$$

for all time $t \geq 0$.

Lemma 2.3. *Assume that (1.4) is satisfied and (v, F) is the solution of system (1.6). Then the following is always true:*

$$\partial_m E_{ij} - \partial_j E_{im} = E_{lj} \partial_l E_{im} - E_{lm} \partial_l E_{ij}, \quad (2.3)$$

for all time $t \geq 0$.

3 Littlewood–Paley theory and Besov spaces

The proof of most of the results presented in this paper requires a dyadic decomposition of Fourier variable (*Littlewood–Paley composition*). Let us briefly explain how it may be built in the case $x \in \mathbb{R}^N$, $N \geq 2$, (see [12, 13, 14]).

Let $\mathcal{S}(\mathbb{R}^N)$ be the Schwarz class. $\varphi(\xi)$ is a smooth function valued in $[0,1]$ such that

$$\text{supp } \varphi \subset \left\{ \frac{3}{4} \leq |\xi| \leq \frac{8}{3} \right\} \quad \text{and} \quad \sum_{q \in \mathbb{Z}} \varphi(2^{-q}\xi) = 1, \quad |\xi| \neq 0.$$

Let $h(x) = (\mathcal{F}^{-1}\varphi)(x)$. For $f \in \mathcal{S}'$ (denote the set of temperate distributives, which is the dual one of \mathcal{S}), we can define the homogeneous dyadic blocks as follows:

$$\Delta_q f(x) := \varphi(2^{-q}D)f(x) = 2^{Nq} \int_{\mathbb{R}^N} h(2^q y) f(x-y) dy, \quad \text{if } q \in \mathbb{Z},$$

where \mathcal{F}^{-1} represents the inverse Fourier transform. Define the low frequency cut-off by

$$S_q f(x) := \sum_{p \leq q-1} \Delta_p f(x) = \chi(2^{-q}D)f(x).$$

The Littlewood–Paley decomposition has nice properties of quasi-orthogonality,

$$\Delta_p \Delta_q f_1 \equiv 0, \quad \text{if } |p-q| \geq 2,$$

and

$$\Delta_q (S_{p-1} f_1 \Delta_p f_2) \equiv 0, \quad \text{if } |p-q| \geq 5.$$

Lemma 3.1 (Bernstein). *Let $k \in \mathbb{N}$ and $0 < R_1 < R_2$. There exists a constant C depending only on R_1, R_2 and N such that for all $1 \leq a \leq b \leq \infty$ and $f \in L^a$, we have*

$$\text{Supp } \mathcal{F}f \subset B(0, R_1 \lambda) \Rightarrow \sup_{|\alpha|=k} \|\partial^\alpha f\|_{L^b} \leq C^{k+1} \lambda^{k+N(\frac{1}{a}-\frac{1}{b})} \|f\|_{L^a};$$

$$\text{Supp } \mathcal{F}f \subset C(0, R_1 \lambda, R_2 \lambda) \Rightarrow C^{-k-1} \lambda^k \|f\|_{L^a} \leq \sup_{|\alpha|=k} \|\partial^\alpha f\|_{L^a} \leq C^{k+1} \lambda^k \|f\|_{L^a}.$$

Here, \mathcal{F} represents the Fourier transform.

The Besov space can be characterized in virtue of the Littlewood–Paley decomposition.

Definition 3.1. Let $1 \leq p \leq \infty$ and $s \in \mathbb{R}$. For $1 \leq r \leq \infty$, the Besov spaces $\dot{B}_{p,r}^s(\mathbb{R}^N)$, $N \geq 2$, are defined by

$$f \in \dot{B}_{p,r}^s(\mathbb{R}^N) \Leftrightarrow \|2^{qs} \|\Delta_q f\|_{L^p(\mathbb{R}^N)}\|_{l_q^r} < \infty$$

and $B^s(\mathbb{R}^N) = \dot{B}_{2,1}^s(\mathbb{R}^N)$.

The definition of $\dot{B}_{p,r}^s(\mathbb{R}^N)$ does not depend on the choice of the Littlewood–Paley decomposition. Let us recall some classical estimates in Sobolev spaces for the product of two functions [6, 15].

Formally, Bony's decomposition is defined by

$$uv = T_u v + T_v u + R(u, v) = T_u v + T_v' u,$$

with

$$T_u v = \sum_{q \in \mathbb{Z}} S_{q-1} u \Delta_q v, \quad R(u, v) = \sum_{q \in \mathbb{Z}} \Delta_q u \tilde{\Delta}_q v, \quad \tilde{\Delta}_q v = \sum_{|q'-q| \leq 1} \Delta_{q'} v.$$

Proposition 3.1. *For any $(s, p, r) \in \mathbb{R} \times [1, \infty]^2$ and $t < 0$, there exists a constant C such that*

$$\|T_u v\|_{\dot{B}_{p,r}^s} \leq C \|u\|_{L^\infty} \|v\|_{\dot{B}_{p,r}^s} \quad \text{and} \quad \|T_u v\|_{\dot{B}_{p,r}^{s+t}} \leq C \|u\|_{\dot{B}_{p,r}^t} \|v\|_{\dot{B}_{p,r}^s}.$$

For any (s_1, p_1, r_1) and (s_2, p_2, r_2) in $\mathbb{R} \times [1, \infty]^2$ and $t < 0$, there exists a constant C such that

- if $s_1 + s_2 > 0$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \leq 1$ and $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} \leq 1$, then

$$\|R(u, v)\|_{\dot{B}_{p,r}^{s_1+s_2}} \leq C \|u\|_{\dot{B}_{p_1,r_1}^{s_1}} \|v\|_{\dot{B}_{p_2,r_2}^{s_2}};$$

- if $s_1 + s_2 = 0$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \leq 1$ and $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} \geq 1$, then

$$\|R(u, v)\|_{\dot{B}_{p,r}^0} \leq C \|u\|_{\dot{B}_{p_1,r_1}^{s_1}} \|v\|_{\dot{B}_{p_2,r_2}^{s_2}}.$$

Proposition 3.2. Let $1 \leq r, p, p_1, p_2 \leq \infty$. Then following inequalities hold true:

$$\|uv\|_{\dot{B}_{p,r}^s} \lesssim \|u\|_{L^\infty} \|v\|_{\dot{B}_{p,r}^s} + \|v\|_{L^\infty} \|u\|_{\dot{B}_{p,r}^s}, \quad \text{if } s > 0, \quad (3.1)$$

$$\|uv\|_{\dot{B}_{p,r}^{s_1+s_2-\frac{N}{p}}} \lesssim \|u\|_{\dot{B}_{p,r}^{s_1}} \|v\|_{\dot{B}_{p,\infty}^{s_2}}, \quad \text{if } s_1, s_2 < \frac{N}{p} \quad \text{and } s_1 + s_2 > 0, \quad (3.2)$$

$$\|uv\|_{\dot{B}_{p,r}^s} \lesssim \|u\|_{\dot{B}_{p,r}^s} \|v\|_{\dot{B}_{p,\infty}^{\frac{N}{p}} \cap L^\infty}, \quad \text{if } |s| < \frac{N}{p}, \quad (3.3)$$

$$\|uv\|_{\dot{B}_{p,\infty}^{-\frac{N}{p}}} \lesssim \|u\|_{\dot{B}_{p,1}^s} \|v\|_{\dot{B}_{p,\infty}^{-s}}, \quad \text{if } s \in (-\frac{N}{p}, \frac{N}{p}], \quad p \geq 2. \quad (3.4)$$

Finally, we state the definition of $\tilde{L}^q(0, T; \dot{B}_{p,r}^s)$, which is first introduced by J.-Y. Chemin and N. Lerner in [8].

Definition 3.2. We denote by $\tilde{L}^q(0, T; \dot{B}_{p,r}^s)$ the space of distributions, which is the completion of $\mathcal{S}(\mathbb{R}^d)$ by the following norm:

$$\|a\|_{\tilde{L}^q(0,T;\dot{B}_{p,r}^s)} = \|2^{sk} \|\Delta_k a\|_{L^q(0,T;L^p(\mathbb{R}^d))}\|_{l_k^r}.$$

Similar results of Propositions 3.1–3.2 in space $\tilde{L}^q(0, T; \dot{B}_{p,r}^s)$ also hold [6, 15].

Denote

$$\|v\|_{\tilde{L}_T^1(\dot{B}_{2,r}^{s+2,s+1})} := \|2^{p(s+1)} \min(R_0, 2^p) \|\Delta_p v\|_{L_T^1(L_x^2)}\|_{l_p^r} \quad (3.5)$$

Lemma 3.2. Let $s > -1$ and $r \in [1, \infty]$, we have

$$\|uv\|_{\tilde{L}_T^1(\dot{B}_{2,r}^{s+2,s+1})} \leq C \|u\|_{L_T^\infty(L_x^\infty)} \|v\|_{\tilde{L}_T^1(\dot{B}_{2,r}^{s+2,s+1})} + C \|v\|_{L_T^\infty(L_x^\infty)} \|u\|_{\tilde{L}_T^1(\dot{B}_{2,r}^{s+2,s+1})}. \quad (3.6)$$

Proof. We write, for $p \in \mathbb{Z}$,

$$\Delta_p T_u v = \sum_{|q-p| \leq 3} \Delta_p (S_{q-1} u \Delta_q v),$$

hence,

$$\|\Delta_p T_u v\|_{L_T^1(L^2)} \leq C \sum_{|q-p| \leq 3} \|u\|_{L_T^\infty(L_x^\infty)} \|\Delta_q v\|_{L_T^1(L^2)},$$

and

$$\|T_u v\|_{\tilde{L}_T^1(\dot{B}_{2,r}^{s+2,s+1})} \leq C \|u\|_{L_T^\infty(L_x^\infty)} \|v\|_{\tilde{L}_T^1(\dot{B}_{2,r}^{s+2,s+1})}. \quad (3.7)$$

Similarly, we have

$$\|T_v u\|_{\tilde{L}_T^1(\dot{B}_{2,r}^{s+2,s+1})} \leq C \|v\|_{L_T^\infty(L_x^\infty)} \|u\|_{\tilde{L}_T^1(\dot{B}_{2,r}^{s+2,s+1})}. \quad (3.8)$$

We write, for $p \in \mathbb{Z}$,

$$\Delta_p R(u, v) = \sum_{q \geq p-2} \Delta_p (\Delta_q u \tilde{\Delta}_q v),$$

hence,

$$2^{p(s+1)} \min(R_0, 2^p) \|\Delta_p R(u, v)\|_{L_T^1(L^2)}$$

$$\begin{aligned}
&\leq C \sum_{q \geq p-2} 2^{(p-q)(s+1)} \frac{\min(R_0, 2^p)}{\min(R_0, 2^q)} 2^{q(s+1)} \min(R_0, 2^q) \|\Delta_q u\|_{L_T^1(L^2)} \|v\|_{L_T^\infty(L_x^\infty)} \\
&\leq C \sum_{q \geq p-2} 2^{(p-q)(s+1)} c_q \|u\|_{\tilde{L}_T^1(\tilde{B}_{2,r}^{s+2,s+1})} \|v\|_{L_T^\infty(L_x^\infty)},
\end{aligned}$$

and

$$\|R(u, v)\|_{\tilde{L}_T^1(\tilde{B}_{2,r}^{s+2,s+1})} \leq C \|v\|_{L_T^\infty(L_x^\infty)} \|u\|_{\tilde{L}_T^1(\tilde{B}_{2,r}^{s+2,s+1})}, \quad (3.9)$$

where we use the fact that $s > -1$. \square

4 A linear model with convection

After the change of function

$$c = \Lambda^{-1} \nabla \cdot E,$$

the system (1.6) reads (1.15). At first, we will study the following mixed linear system

$$\begin{cases} \nabla \cdot v = \nabla \cdot c = 0, & x \in \mathbb{R}^N, \ N \geq 2, \\ v_{it} + T_u \cdot \nabla v_i + \partial_i p - \mu \Delta v_i - \Lambda c_i = G_i, \\ c_t + T_u \cdot \nabla c + \Lambda v = L, \\ (v, c)(0, x) = (v_0, c_0)(x), \end{cases} \quad (4.1)$$

where $\operatorname{div} u = 0$ and $c_0 = \Lambda^{-1} \nabla \cdot E_0$.

Using the similar arguments as that in [12] (Proposition 2.3), we can obtain the following proposition and omit the details. The main different is that there is a pressure term ∇p in (4.1)₂. Using that fact that $\nabla \cdot v = \nabla \cdot c = 0$, we have $\int v \cdot \nabla p = \int c \cdot \nabla p = 0$, and obtain the following proposition.

Proposition 4.1. *Let (v, c) be a solution of (4.1) on $[0, T]$, $\rho \in \mathbb{R}$ and $\tilde{U}(t) = \int_0^t \|\nabla u(\tau)\|_{L^\infty} d\tau$. The following estimate holds:*

$$\begin{aligned}
&\|(v, c)^l\|_{\tilde{L}_t^\infty(\dot{B}_{2,r}^\rho) \cap \tilde{L}_t^1(\dot{B}_{2,r}^{\rho+2})} + \|c^h\|_{\tilde{L}_t^\infty(\dot{B}_{2,r}^{\rho+1}) \cap \tilde{L}_t^1(\dot{B}_{2,r}^{\rho+1})} + \|v^h\|_{\tilde{L}_t^\infty(\dot{B}_{2,r}^\rho) \cap \tilde{L}_t^1(\dot{B}_{2,r}^{\rho+2})} \\
&\leq C e^{C\tilde{U}(t)} (\|(v_0, c_0)^l\|_{\dot{B}_{2,r}^\rho} + \|c_0^h\|_{\dot{B}_{2,r}^{\rho+1}} + \|v_0^h\|_{\dot{B}_{2,r}^\rho} + \|(L, G)^l\|_{\tilde{L}_t^1(\dot{B}_{2,r}^\rho)} \\
&\quad + \|L^h\|_{\tilde{L}_t^1(\dot{B}_{2,r}^{\rho+1})} + \|G^h\|_{\tilde{L}_t^1(\dot{B}_{2,r}^\rho)}),
\end{aligned}$$

where C depends only on N and ρ .

Then, we will study the following mixed linear system

$$\begin{cases} \nabla \cdot v = \nabla \cdot c = 0, & x \in \mathbb{R}^N, \ N \geq 2, \\ v_t - \mu \Delta v - \Lambda c = 0, \\ c_t + \Lambda v = 0, \\ (v, c)(0, x) = (v_0, c_0)(x). \end{cases} \quad (4.2)$$

Using the similar arguments as that in [7] (Lemma 4.1–4.2, Proposition 4.4), we have the following lemma.

Lemma 4.1. *Let \mathcal{G} be the Green matrix of the following system*

$$\begin{cases} c_t + \Lambda v = 0, \\ v_t - \mu \Delta v - \Lambda c = 0, \end{cases} \quad (4.3)$$

then, we have the following explicit expression for $\hat{\mathcal{G}}$:

$$\hat{\mathcal{G}} = \begin{bmatrix} \frac{\lambda_+ e^{\lambda_- t} - \lambda_- e^{\lambda_+ t}}{\lambda_+ - \lambda_-} I_{N \times N} & -\frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-} |\xi| I_{N \times N} \\ -\frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-} |\xi| I_{N \times N} & \frac{\lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t}}{\lambda_+ - \lambda_-} I_{N \times N} \end{bmatrix}, \quad (4.4)$$

where

$$\lambda_{\pm} = -\frac{1}{2} \mu |\xi|^2 \pm \frac{1}{2} \sqrt{\mu^2 |\xi|^4 - 4 |\xi|^2}.$$

Lemma 4.2. *There exist positive constant R and ϑ depending on μ such that*

$$|\partial_\xi^\alpha \hat{\mathcal{G}}(\xi, t)| \leq C_{|\alpha|} e^{-\vartheta|\xi|^2 t} (1 + |\xi|)^\alpha (1 + t)^{|\alpha|}, \quad \forall |\xi| \leq R, \quad (4.5)$$

$$\begin{aligned} \hat{\mathcal{G}}(\xi, t) &= e^{-\mu^{-1}t} \begin{bmatrix} I_{N \times N} & 0 \\ 0 & 0 \end{bmatrix} + e^{-\mu|\xi|^2 t} \begin{bmatrix} 0 & 0 \\ 0 & I_{N \times N} \end{bmatrix} \\ &+ \hat{\mathcal{G}}^1(\xi, t) \begin{bmatrix} 0 & I_{N \times N} \\ I_{N \times N} & 0 \end{bmatrix} + \hat{\mathcal{G}}^2(\xi, t) \begin{bmatrix} I_{N \times N} & 0 \\ 0 & I_{N \times N} \end{bmatrix}, \quad \forall |\xi| \geq R, \end{aligned} \quad (4.6)$$

where $\hat{\mathcal{G}}^1$ and $\hat{\mathcal{G}}^2$ satisfy the estimates

$$|\partial_\xi^\alpha \hat{\mathcal{G}}^1| \leq C|\xi|^{-|\alpha|-1} (e^{-\frac{t}{2\mu}} + e^{-\vartheta|\xi|^2 t}), \quad (4.7)$$

$$|\partial_\xi^\alpha \hat{\mathcal{G}}^2| \leq C|\xi|^{-|\alpha|-2} (e^{-\frac{t}{2\mu}} + e^{-\vartheta|\xi|^2 t}). \quad (4.8)$$

Proposition 4.2. *Let \mathcal{C} be a ring centered at 0 in \mathbb{R}^N . Then there exist positive constants R_0 , C , ϑ depending on μ such that, if $\text{supp } \hat{u} \subset \lambda\mathcal{C}$, then we have*

(a) if $\lambda \leq R_0$, then

$$\|\mathcal{G} * u\|_{L^2} \leq C e^{-\vartheta\lambda^2 t} \|u\|_{L^2}; \quad (4.9)$$

(b) if $b \leq \lambda \leq R_0$, then for any $1 \leq p \leq \infty$,

$$\|\mathcal{G} * u\|_{L^p} \leq C(1 + b^{-N-1}) e^{-\vartheta\lambda^2 t} \|u\|_{L^p}; \quad (4.10)$$

(c) if $\lambda > R_0$, then for any $1 \leq p \leq \infty$,

$$\|\mathcal{G}^1 * u\|_{L^p} \leq C\lambda^{-1} e^{-\vartheta t} \|u\|_{L^p}, \quad (4.11)$$

$$\|\mathcal{G}^2 * u\|_{L^p} \leq C\lambda^{-2} e^{-\vartheta t} \|u\|_{L^p}. \quad (4.12)$$

From Proposition 4.2, or using the similar arguments as that in [6] (lemma 2), we can obtain the following lemma.

Lemma 4.3. *Let (v, c) be a solution of (4.2). There exist two constants ϑ and C such that, if $2^j > R_0$, then for all $p \in [1, \infty]$,*

$$\|\Delta_j v(t)\|_{L^p} \leq C \left(2^{-j} e^{-\vartheta t} \|\Delta_j c_0\|_{L^p} + \left(e^{-\vartheta t 2^{2j}} + 2^{-2j} e^{-\vartheta t} \right) \|\Delta_j v_0\|_{L^p} \right), \quad (4.13)$$

$$\|\Delta_j c(t)\|_{L^p} \leq C e^{-\vartheta t} \left(\|\Delta_j c_0\|_{L^p} + 2^{-j} \|\Delta_j v_0\|_{L^p} \right). \quad (4.14)$$

For all $m \geq 1$, there exist two constants α and C such that, if $2^j \leq R_0$ then

$$e^{\vartheta t 2^{2j}} \|(\Delta_j c, \Delta_j v)(t)\|_{L^2} \leq C \|(\Delta_j c_0, \Delta_j v_0)\|_{L^2}. \quad (4.15)$$

5 A priori estimates

Assume that v, E be a solution of (1.6). Letting

$$X_{p,0} = Y_{s,0} + Z_{p,0}, \quad (5.1)$$

$$Y_{s,0} = \|E_0^l\|_{\dot{B}_{2,r}^s} + \|E_0^h\|_{\dot{B}_{2,r}^{s+1}} + \|v_0\|_{\dot{B}_{2,r}^s}, \quad (5.2)$$

$$Z_{p,0} = \|E_0^h\|_{\dot{B}_{p,1}^{\frac{N}{p}}} + \|v_0^h\|_{\dot{B}_{p,1}^{\frac{N}{p}-1}}, \quad (5.3)$$

and

$$X_p(t) = Y_s(t) + Z_p(t), \quad (5.4)$$

$$Y_s(t) = \|E^l\|_{\tilde{L}_t^2(\dot{B}_{2,r}^{s+1}) \cap \tilde{L}_t^1(\dot{B}_{2,r}^{s+2})} + \|E\|_{\tilde{L}_t^\infty(\dot{B}_{2,r}^{s+1})} + \|E^h\|_{\tilde{L}_t^1(\dot{B}_{2,r}^{s+1})} + \|v\|_{\tilde{L}_t^\infty(\dot{B}_{2,r}^s) \cap \tilde{L}_t^1(\dot{B}_{2,r}^{s+2})}, \quad (5.5)$$

$$Z_p(t) = \|E^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{N}{p}})} + \|E^h\|_{\tilde{L}_t^1(\dot{B}_{p,1}^{\frac{N}{p}})} + \|v^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{N}{p}-1}) \cap \tilde{L}_t^1(\dot{B}_{p,1}^{\frac{N}{p}+1})}. \quad (5.6)$$

In this section, we will prove the following **Claim 1**:

There exist two positive constants λ_1 and C_0 such that, if

$$X_{p_2}(t) \leq \lambda_1, \text{ for all } t \in [0, T], \quad (5.7)$$

then

$$X_{p_i}(t) \leq C_0 X_{p_i,0}, \text{ for all } i = 1, 2 \text{ } t \in [0, T]. \quad (5.8)$$

Lemma 5.1. *Under the condition (5.7), we have*

$$Y_s(t) \leq C_1 Y_{s,0}, \text{ for all } t \in [0, T]. \quad (5.9)$$

Proof. Let

$$c = \Lambda^{-1} \nabla \cdot E,$$

where $\Lambda^s f = \mathcal{F}^{-1}(|\xi|^s \hat{f})$. Then, the system (1.6) reads

$$\begin{cases} \nabla \cdot v = \nabla \cdot c = 0, & x \in \mathbb{R}^N, \quad N \geq 2, \\ v_{it} + T_v \cdot \nabla v_i + \partial_i p - \mu \Delta v_i - \Lambda c_i = G_i := -\partial_j T'_{v_i} v_j + E_{jk} \partial_j E_{ik}, \\ c_t + T_v \cdot \nabla c + \Lambda v = L := -\partial_j T'_c v_j + \Lambda^{-1} \nabla \cdot (\nabla v E) - [\Lambda^{-1} \nabla \cdot, v] \nabla E, \\ \Delta E_{ij} = \Lambda \partial_j c_i + \partial_k (\partial_k E_{ij} - \partial_j E_{ik}), \\ (v, c)(0, x) = (v_0, \Lambda^{-1} \nabla \cdot E_0)(x). \end{cases} \quad (5.10)$$

From Proposition 4.1, we have

$$\begin{aligned} & \|(v, c)^l\|_{\tilde{L}_t^\infty(\dot{B}_{2,r}^s) \cap \tilde{L}_t^1(\dot{B}_{2,r}^{s+2})} + \|c^h\|_{\tilde{L}_t^\infty(\dot{B}_{2,r}^{s+1}) \cap \tilde{L}_t^1(\dot{B}_{2,r}^{s+1})} + \|v^h\|_{\tilde{L}_t^\infty(\dot{B}_{2,r}^s) \cap \tilde{L}_t^1(\dot{B}_{2,r}^{s+2})} \\ & \leq C e^{C\tilde{U}(t)} (\|(v_0, c_0)^l\|_{\dot{B}_{2,r}^s} + \|c_0^h\|_{\dot{B}_{2,r}^{s+1}} + \|v_0^h\|_{\dot{B}_{2,r}^s} + \|L^l\|_{\tilde{L}_t^1(\dot{B}_{2,r}^s)} \\ & \quad + \|L^h\|_{\tilde{L}_t^1(\dot{B}_{2,r}^{s+1})} + \|G\|_{\tilde{L}_t^1(\dot{B}_{2,r}^s)}) \\ & \leq C (Y_{s,0} + \|L^l\|_{\tilde{L}_t^1(\dot{B}_{2,r}^s)} + \|L^h\|_{\tilde{L}_t^1(\dot{B}_{2,r}^{s+1})} + \|G\|_{\tilde{L}_t^1(\dot{B}_{2,r}^s)}), \end{aligned} \quad (5.11)$$

where $\tilde{U}(t) = \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau \leq C X_{p_2}(t) \leq C$, when $\lambda_1 \leq 1$. From Proposition 3.1 and $s > -1$, we have, for all $t \in [0, T]$,

$$\begin{aligned} \|\partial_j T'_{v_i} v_j\|_{\tilde{L}_t^1(\dot{B}_{2,r}^s)} & \leq \|T_{\partial_j v_i} v_j\|_{\tilde{L}_t^1(\dot{B}_{2,r}^s)} + \|R(v_i, v_j)\|_{\tilde{L}_t^1(\dot{B}_{2,r}^{s+1})} \\ & \leq C \|v\|_{L_t^1(\dot{B}_{p_2,1}^{\frac{N}{p_2}+1})} \|v\|_{\tilde{L}_t^\infty(\dot{B}_{2,r}^s)} \leq C \lambda_1 \|v\|_{\tilde{L}_t^\infty(\dot{B}_{2,r}^s)}. \end{aligned} \quad (5.12)$$

From (3.1) and $\nabla \cdot E^\top = 0$, we have under the assumption that $s+1 > 0$, $s+1 < \frac{N}{2}$ (if $r > 1$) or $s+1 \leq \frac{N}{2}$ (if $r = 1$),

$$\|E_{jk} \partial_j E_{ik}\|_{\tilde{L}_t^1(\dot{B}_{2,r}^s)} = \|\partial_j (E_{jk} E_{ik})\|_{\tilde{L}_t^1(\dot{B}_{2,r}^s)} \leq C \|E\|_{L_t^2(L^\infty)} \|E\|_{\tilde{L}_t^2(\dot{B}_{2,r}^{s+1})} \leq C \lambda_1 \|E\|_{\tilde{L}_t^2(\dot{B}_{2,r}^{s+1})}. \quad (5.13)$$

From (5.12)–(5.13), we have

$$\|G\|_{\tilde{L}_t^1(\dot{B}_{2,r}^s)} \leq C \lambda_1 (\|v\|_{\tilde{L}_t^\infty(\dot{B}_{2,r}^s)} + \|E\|_{\tilde{L}_t^2(\dot{B}_{2,r}^{s+1})}). \quad (5.14)$$

From Proposition 3.1, under the assumption that $s+1 > 0$, $s+1 < \frac{N}{2}$ (if $r > 1$) or $s+1 \leq \frac{N}{2}$ (if $r = 1$), we have, for all $t \in [0, T]$,

$$\|(\partial_j T'_c v_j)^l\|_{\tilde{L}_t^1(\dot{B}_{2,r}^s)} \leq C \|(T'_c v)^l\|_{\tilde{L}_t^1(\dot{B}_{2,r}^{s+1})} \leq C \|c\|_{L_t^2(L^\infty)} \|v\|_{\tilde{L}_t^2(\dot{B}_{2,r}^{s+1})} \leq C \lambda_1 \|v\|_{\tilde{L}_t^2(\dot{B}_{2,r}^{s+1})}, \quad (5.15)$$

$$\|(\partial_j T'_c v_j)^h\|_{\tilde{L}_t^1(\dot{B}_{2,r}^{s+1})} \leq C \|c\|_{L_t^\infty(L^\infty)} \|v\|_{\tilde{L}_t^1(\dot{B}_{2,r}^{s+2})} \leq C \lambda_1 \|v\|_{\tilde{L}_t^1(\dot{B}_{2,r}^{s+2})}. \quad (5.16)$$

where we use the following estimates

$$\|c\|_{L^\infty} \leq \|c^l\|_{L^\infty} + \|c^h\|_{L^\infty} \leq C \left(\|c^l\|_{\dot{B}_{2,1}^{\frac{N}{2}}} + \|c^h\|_{\dot{B}_{p_2,1}^{\frac{N}{p_2}}} \right) \leq C \left(\|E^l\|_{\dot{B}_{2,1}^{\frac{N}{2}}} + \|E^h\|_{\dot{B}_{p_2,1}^{\frac{N}{p_2}}} \right)$$

and

$$\|c\|_{L_t^2(L^\infty)} + \|c\|_{L_t^\infty(L^\infty)} \leq C X_{p_2}(t) \leq C \lambda_1.$$

Similarly, from Proposition 3.1, we have,

$$\|(T_E \nabla v)^l\|_{\tilde{L}_t^1(\dot{B}_{2,r}^s)} \leq C \|E\|_{L_t^2(L^\infty)} \|\nabla v\|_{\tilde{L}_t^2(\dot{B}_{2,r}^s)} \leq C \lambda_1 \|\nabla v\|_{\tilde{L}_t^2(\dot{B}_{2,r}^s)}, \quad (5.17)$$

$$\|(T_E \nabla v)^h\|_{\tilde{L}_t^1(\dot{B}_{2,r}^{s+1})} \leq C \|E\|_{L_t^\infty(L^\infty)} \|\nabla v\|_{\tilde{L}_t^1(\dot{B}_{2,r}^{s+1})} \leq C \lambda_1 \|\nabla v\|_{\tilde{L}_t^1(\dot{B}_{2,r}^{s+1})}, \quad (5.18)$$

$$\begin{aligned} \|(T_{\nabla v} E)^l\|_{\tilde{L}_t^1(\dot{B}_{2,r}^s)} &\leq C \|\nabla v\|_{L_t^2(\dot{B}_{\infty,\infty}^{-1})} \|E\|_{\tilde{L}_t^2(\dot{B}_{2,r}^{s+1})} \\ &\leq C \lambda_1 \|E\|_{\tilde{L}_t^2(\dot{B}_{2,r}^{s+1})}, \end{aligned} \quad (5.19)$$

$$\begin{aligned} \|(T_{\nabla v} E)^h\|_{\tilde{L}_t^1(\dot{B}_{2,r}^{s+1})} &\leq C \|\nabla v\|_{L_t^1(L^\infty)} \|E\|_{\tilde{L}_t^\infty(\dot{B}_{2,r}^{s+1})} \\ &\leq C \lambda_1 (\|E^l\|_{\tilde{L}_t^\infty(\dot{B}_{2,r}^s)} + \|E^h\|_{\tilde{L}_t^\infty(\dot{B}_{2,r}^{s+1})}), \end{aligned} \quad (5.20)$$

$$\begin{aligned} \|(R(E_{jk}, \partial_j v_i))^l\|_{\tilde{L}_t^1(\dot{B}_{2,r}^s)} &= \|\partial_j (R(E_{jk}, v_i))^l\|_{\tilde{L}_t^1(\dot{B}_{2,r}^s)} \\ &\leq \|(R(E, v))^l\|_{\tilde{L}_t^1(\dot{B}_{\frac{2p_2}{2+p_2},r}^{s+\frac{N}{p_2}+1})} \\ &\leq C \|v\|_{L_t^2(\dot{B}_{p_2,\infty}^{\frac{N}{p_2}})} \|E\|_{\tilde{L}_t^2(\dot{B}_{2,r}^{s+1})} \\ &\leq C \lambda_1 \|E\|_{\tilde{L}_t^2(\dot{B}_{2,r}^{s+1})}, \end{aligned} \quad (5.21)$$

$$\begin{aligned} \|(R(E, \nabla v))^h\|_{\tilde{L}_t^1(\dot{B}_{2,r}^{s+1})} &\leq \|(R(E, \nabla v))^l\|_{\tilde{L}_t^1(\dot{B}_{\frac{2p_2}{2+p_2},r}^{s+\frac{N}{p_2}+1})} \\ &\leq C \|\nabla v\|_{L_t^1(\dot{B}_{p_2,\infty}^{\frac{N}{p_2}})} \|E\|_{\tilde{L}_t^\infty(\dot{B}_{2,r}^{s+1})} \\ &\leq C \lambda_1 (\|E^l\|_{\tilde{L}_t^\infty(\dot{B}_{2,r}^s)} + \|E^h\|_{\tilde{L}_t^\infty(\dot{B}_{2,r}^{s+1})}), \end{aligned} \quad (5.22)$$

where we use the following estimates

$$\|\nabla v\|_{L_t^2(\dot{B}_{\infty,\infty}^{-1})} \leq C \|\nabla v\|_{L_t^2(\dot{B}_{p_2,1}^{\frac{N}{p_2}-1})} \leq C X_{p_2}(t) \leq C \lambda_1,$$

$$\|\nabla v\|_{L_t^1(L^\infty)} \leq C \|\nabla v\|_{L_t^1(\dot{B}_{p_2,1}^{\frac{N}{p_2}})} \leq C X_{p_2}(t) \leq C \lambda_1.$$

From (5.17)–(5.22), we obtain

$$\|(\Lambda^{-1} \nabla \cdot (\nabla v E))^l\|_{\tilde{L}_t^1(\dot{B}_{2,r}^s)} \leq C \lambda_1 (\|\nabla v\|_{\tilde{L}_t^2(\dot{B}_{2,r}^s)} + \|E\|_{\tilde{L}_t^2(\dot{B}_{2,r}^{s+1})}), \quad (5.23)$$

$$\|(\Lambda^{-1} \nabla \cdot (\nabla v E))^h\|_{\tilde{L}_t^1(\dot{B}_{2,r}^{s+1})} \leq C \lambda_1 (\|\nabla v\|_{\tilde{L}_t^1(\dot{B}_{2,r}^{s+1})} + \|E^l\|_{\tilde{L}_t^\infty(\dot{B}_{2,r}^s)} + \|E^h\|_{\tilde{L}_t^\infty(\dot{B}_{2,r}^{s+1})}). \quad (5.24)$$

Similar, using the classical commutator estimate (Lemma 2.97 in [2]), we obtain

$$\|([\Lambda^{-1} \nabla \cdot, v] \nabla E)^l\|_{\tilde{L}_t^1(\dot{B}_{2,r}^s)} \leq C \lambda_1 (\|\nabla v\|_{\tilde{L}_t^2(\dot{B}_{2,r}^s)} + \|E\|_{\tilde{L}_t^2(\dot{B}_{2,r}^{s+1})}), \quad (5.25)$$

$$\|([\Lambda^{-1}\nabla\cdot, v]\nabla E)^h\|_{\tilde{L}_t^1(\dot{B}_{2,r}^{s+1})} \leq C\lambda_1(\|\nabla v\|_{\tilde{L}_t^1(\dot{B}_{2,r}^{s+1})} + \|E^l\|_{\tilde{L}_t^\infty(\dot{B}_{2,r}^s)} + \|E^h\|_{\tilde{L}_t^\infty(\dot{B}_{2,r}^{s+1})}). \quad (5.26)$$

From (5.15)–(5.16) and (5.23)–(5.26), we get

$$\|L^l\|_{\tilde{L}_t^1(\dot{B}_{2,r}^s)} \leq C\lambda_1(\|v\|_{\tilde{L}_t^2(\dot{B}_{2,r}^{s+1})} + \|E\|_{\tilde{L}_t^2(\dot{B}_{2,r}^{s+1})}), \quad (5.27)$$

$$\|L^h\|_{\tilde{L}_t^1(\dot{B}_{2,r}^{s+1})} \leq C\lambda_1(\|v\|_{\tilde{L}_t^1(\dot{B}_{2,r}^{s+2})} + \|E^l\|_{\tilde{L}_t^\infty(\dot{B}_{2,r}^s)} + \|E^h\|_{\tilde{L}_t^\infty(\dot{B}_{2,r}^{s+1})}). \quad (5.28)$$

From (5.11), (5.14) and (5.27)–(5.28), we have

$$\begin{aligned} & \|(v, c)^l\|_{\tilde{L}_t^\infty(\dot{B}_{2,r}^s) \cap \tilde{L}_t^1(\dot{B}_{2,r}^{s+2})} + \|c^h\|_{\tilde{L}_t^\infty(\dot{B}_{2,r}^{s+1}) \cap \tilde{L}_t^1(\dot{B}_{2,r}^{s+1})} + \|v^h\|_{\tilde{L}_t^\infty(\dot{B}_{2,r}^s) \cap \tilde{L}_t^1(\dot{B}_{2,r}^{s+2})} \\ & \leq CY_{s,0} + C\lambda_1 Y_s(t). \end{aligned} \quad (5.29)$$

From Lemmas 2.2–2.3, we have

$$\partial_m E_{ij} - \partial_j E_{im} = E_{lj} \partial_l E_{im} - E_{lm} \partial_l E_{ij} = \partial_l (E_{lj} E_{im} - E_{lm} E_{ij}). \quad (5.30)$$

and from (5.10)₄, Proposition 3.2, Lemma 3.2,

$$\begin{aligned} & \|E\|_{\tilde{L}_t^\infty(\dot{B}_{2,r}^{s+1})} + \|E^l\|_{\tilde{L}_t^2(\dot{B}_{2,r}^{s+1}) \cap \tilde{L}_t^1(\dot{B}_{2,r}^{s+2})} + \|E^h\|_{\tilde{L}_t^1(\dot{B}_{2,r}^{s+1})} \\ & \leq C\|c\|_{\tilde{L}_t^\infty(\dot{B}_{2,r}^{s+1})} + C\|c^l\|_{\tilde{L}_t^2(\dot{B}_{2,r}^{s+1}) \cap \tilde{L}_t^1(\dot{B}_{2,r}^{s+2})} + C\|c^h\|_{\tilde{L}_t^1(\dot{B}_{2,r}^{s+1})} \\ & \quad + C\|E^2\|_{\tilde{L}_t^\infty(\dot{B}_{2,r}^{s+1})} + C\|(E^2)^l\|_{\tilde{L}_t^2(\dot{B}_{2,r}^{s+1}) \cap \tilde{L}_t^1(\dot{B}_{2,r}^{s+2})} + C\|(E^2)^h\|_{\tilde{L}_t^1(\dot{B}_{2,r}^{s+1})} \\ & \leq C\|c\|_{\tilde{L}_t^\infty(\dot{B}_{2,r}^{s+1})} + C\|c^l\|_{\tilde{L}_t^2(\dot{B}_{2,r}^{s+1}) \cap \tilde{L}_t^1(\dot{B}_{2,r}^{s+2})} + C\|c^h\|_{\tilde{L}_t^1(\dot{B}_{2,r}^{s+1})} \\ & \quad + C\|E\|_{L_t^\infty(\dot{B}_{p_2,1}^{\frac{N}{2}})} (\|E\|_{\tilde{L}_t^\infty(\dot{B}_{2,r}^{s+1})} + \|E^l\|_{\tilde{L}_t^2(\dot{B}_{2,r}^{s+1}) \cap \tilde{L}_t^1(\dot{B}_{2,r}^{s+2})} + \|E^h\|_{\tilde{L}_t^1(\dot{B}_{2,r}^{s+1})}) \\ & \leq C\|c\|_{\tilde{L}_t^\infty(\dot{B}_{2,r}^{s+1})} + C\|c^l\|_{\tilde{L}_t^2(\dot{B}_{2,r}^{s+1}) \cap \tilde{L}_t^1(\dot{B}_{2,r}^{s+2})} + C\|c^h\|_{\tilde{L}_t^1(\dot{B}_{2,r}^{s+1})} \\ & \quad + C\lambda_1(\|E\|_{\tilde{L}_t^\infty(\dot{B}_{2,r}^{s+1})} + \|E^l\|_{\tilde{L}_t^2(\dot{B}_{2,r}^{s+1}) \cap \tilde{L}_t^1(\dot{B}_{2,r}^{s+2})} + \|E^h\|_{\tilde{L}_t^1(\dot{B}_{2,r}^{s+1})}). \end{aligned}$$

When λ_1 is small enough, $C\lambda_1 < \frac{1}{2}$, we have

$$\begin{aligned} & \|E\|_{\tilde{L}_t^\infty(\dot{B}_{2,r}^{s+1})} + \|E^l\|_{\tilde{L}_t^2(\dot{B}_{2,r}^{s+1}) \cap \tilde{L}_t^1(\dot{B}_{2,r}^{s+2})} + \|E^h\|_{\tilde{L}_t^1(\dot{B}_{2,r}^{s+1})} \\ & \leq C\|c\|_{\tilde{L}_t^\infty(\dot{B}_{2,r}^{s+1})} + C\|c^l\|_{\tilde{L}_t^2(\dot{B}_{2,r}^{s+1}) \cap \tilde{L}_t^1(\dot{B}_{2,r}^{s+2})} + C\|c^h\|_{\tilde{L}_t^1(\dot{B}_{2,r}^{s+1})}. \end{aligned} \quad (5.31)$$

From (5.29) and (5.31), we get

$$Y_s(t) \leq CY_{s,0} + C\lambda_1 Y_s(t).$$

When λ_1 is small enough, $C\lambda_1 < \frac{1}{2}$, we obtain $Y_s(t) \leq CY_{s,0}$. \square

Lemma 5.2. *Under the condition (5.7), we have*

$$Z_p(t) \leq C_2(Z_{p,0} + Y_{s,0}), \text{ for all } p = p_1, p_2 \text{ } t \in [0, T]. \quad (5.32)$$

Proof. Applying operator Δ_q to (1.6), we obtain the following system for $(v_q, E_q) := (\Delta_q v, \Delta_q E)$,

$$\begin{cases} \nabla \cdot v_q = 0, & x \in \mathbb{R}^N, \quad N \geq 2, \\ \partial_t v_{q,i} + S_{q-1} v \cdot \nabla v_{q,i} - \mu \Delta v_{q,i} - \partial_j E_{q,ij} = G_{q,i} \\ := S_{q-1} v \cdot \nabla v_{q,i} - \Delta_q (T_v \cdot \nabla v_i) + \Delta_q (-T'_{\nabla v_i} \cdot v - \partial_i p + E_{jk} \partial_j E_{ik}), \\ \partial_t E_q + S_{q-1} v \cdot \nabla E_q - \nabla v_q = L_q \\ := \partial_i (S_{q-1} v_i E_q - \Delta_q T_{v_i} E) - \Delta_q \partial_i T'_E v_i + \Delta_q (\nabla v E), \\ (v_q, E_q)(0, x) = (\Delta_q v_0, \Delta_q E_0)(x). \end{cases} \quad (5.33)$$

Since

$$L_q = \Delta_q L + \partial_i \left(\sum_{q' \sim q} (S_{q-1} v_i - S_{q'-1} v_i) \Delta_q \Delta_{q'} E + [S_{q'-1} v_i, \Delta_q] \Delta_{q'} E \right),$$

where $L := \nabla v E - \partial_i T'_E v_i$, using Lemma 3.1 and the classical commutator estimate (Lemma 2.97 in [2]), we have

$$\|\nabla L_q\|_{L^p} \leq \|\nabla \Delta_q L\|_{L^p} + C \|\nabla v\|_{L^\infty} \sum_{q' \sim q} \|\nabla \Delta_{q'} E\|_{L^p}. \quad (5.34)$$

Similar, we get

$$\|G_q\|_{L^p} \leq \|\Delta_q G\|_{L^p} + C \|\nabla v\|_{L^\infty} \sum_{q' \sim q} \|\Delta_{q'} v\|_{L^p}, \quad (5.35)$$

where $G := -T'_{\nabla v_i} \cdot v - \partial_i p + E_{jk} \partial_j E_{ik}$.

In order to handle the convection terms, one may perform the Lagrangian change of variable $(\tau, x) = (t, \psi_q(t, y))$, where ψ_q stands for the flow of $S_{q-1}v$. Let $\phi_q := \psi_q^{-1}$, $\tilde{f}_q = f_q \circ \psi_q$. Then, $(\tilde{v}_q, \tilde{E}_q) := (v_q \circ \psi_q, E_q \circ \psi_q)$ satisfies

$$\begin{cases} \nabla \cdot v_q = 0, & x \in \mathbb{R}^N, \quad N \geq 2, \\ \partial_t \tilde{v}_q - \mu \Delta \tilde{v}_q - \nabla \cdot \tilde{E}_q = \tilde{G}_q + R_q^1 + R_q^2, \\ \partial_t \tilde{E}_q - \nabla \tilde{v}_q = \tilde{L}_q + R_q^3, \\ (\tilde{v}_q, \tilde{E}_q)(0, x) = (\Delta_q v_0, \Delta_q E_0)(x), \end{cases} \quad (5.36)$$

where

$$R_{q,i}^1(t, x) := \partial_k \tilde{E}_{q,ij}(t, x) (\partial_j \phi_{q,k}(t, \psi_q(t, x)) - \delta_{jk}),$$

$$\begin{aligned} R_{q,i}^2 &:= \mu (\partial_j \phi_{q,k}(t, \psi_q(t, x)) - \delta_{jk}) \partial_k \partial_l \tilde{v}_{q,i}(t, x) \cdot \partial_j \phi_{q,l}(t, \psi_q(t, x)) \\ &\quad + \mu \partial_j \partial_l \tilde{v}_{q,i}(t, x) (\partial_j \phi_{q,l}(t, \psi_q(t, x)) - \delta_{lj}) \\ &\quad + \mu \partial_j \tilde{v}_{q,i}(t, x) \Delta \phi_{q,j}(t, \psi_q(t, x)). \end{aligned}$$

$$R_{q,jk}^3 := \partial_l \tilde{v}_{qj}(t, x) \cdot (\partial_k \phi_{q,l}(t, \psi_q(t, x)) - \delta_{lk}).$$

Let

$$\tilde{c}_q = \Lambda^{-1} \nabla \cdot \tilde{E}_q,$$

where $\Lambda^s f = \mathcal{F}^{-1}(|\xi|^s \hat{f})$. Then, the system (5.36) reads

$$\begin{cases} \partial_t \tilde{v}_{q,i} - \mu \Delta \tilde{v}_{q,i} - \Lambda \tilde{c}_{q,i} = \tilde{G}_q + R_q^1 + R_q^2, \\ \partial_t \tilde{c}_q + \Lambda \tilde{v}_q = \Lambda^{-1} \nabla \cdot \tilde{L}_q + \Lambda^{-1} \nabla \cdot R_q^3, \\ (\tilde{v}_q, \tilde{c}_q)(0, x) = (\Delta_q v_0, \Lambda^{-1} \nabla \cdot \Delta_q E_0)(x). \end{cases} \quad (5.37)$$

Then we may write

$$\begin{pmatrix} \Delta_j \tilde{c}_q(t) \\ \Delta_j \tilde{v}_q(t) \end{pmatrix} = \mathcal{G}(\cdot, t) * \begin{pmatrix} \Delta_j \Lambda^{-1} \nabla \cdot \Delta_q E_0 \\ \Delta_j \Delta_q v_0 \end{pmatrix} + \int_0^t \mathcal{G}(\cdot, t - \tau) * \begin{pmatrix} \Delta_j \Lambda^{-1} \nabla \cdot (\tilde{L}_q + R_q^3) \\ \Delta_j (\tilde{G}_q + R_q^1 + R_q^2) \end{pmatrix} d\tau$$

From Lemma 4.3, we have, for all $2^j > R_0$, then for all $p \in [1, \infty]$,

$$\begin{aligned} & \|\Delta_j \tilde{v}_q(t)\|_{L^p} \\ & \leq C 2^{-2j} e^{-\vartheta t} \|\Delta_j \Delta_q \nabla E_0\|_{L^p} \\ & \quad + C \left(e^{-\vartheta 2^{2j} t} + 2^{-2j} e^{-\vartheta t} \right) \|\Delta_j \Delta_q u_0\|_{L^p} \\ & \quad + C \int_0^t (2^{-2j} e^{-\vartheta(t-\tau)} (\|\Delta_j \nabla \tilde{L}_q\|_{L^p} + \|\Delta_j \nabla R_q^3\|_{L^p}) \\ & \quad + (e^{-\vartheta 2^{2j}(t-\tau)} + 2^{-2j} e^{-\vartheta(t-\tau)}) (\|\Delta_j \tilde{G}_q\|_{L^p} + \|\Delta_j R_q^1\|_{L^p} + \|\Delta_j R_q^2\|_{L^p})) d\tau, \end{aligned} \quad (5.38)$$

$$\begin{aligned} & \|\Delta_j \nabla \tilde{c}_q\|_{L^p} \\ & \leq C e^{-\vartheta t} (\|\Delta_j \Delta_q \nabla E_0\|_{L^p} + \|\Delta_j \Delta_q v_0\|_{L^p}) \\ & \quad + C \int_0^t e^{-\vartheta(t-\tau)} \left(\|\Delta_j \nabla \tilde{L}_q\|_{L^p} + \|\Delta_j \nabla R_q^3\|_{L^p} + \|\Delta_j \tilde{G}_q\|_{L^p} \right. \end{aligned}$$

$$+\|\Delta_j R_q^1\|_{L^p} + \|\Delta_j R_q^2\|_{L^p}) d\tau. \quad (5.39)$$

We now have to find suitable bounds for the right-hand side of (5.38)–(5.39). In what follows we shall freely use the following estimate (Proposition 8 in [6]):

$$\|g \circ \psi_q\|_{L^p} = \|g\|_{L^p}, \text{ for all function } g \in L^p, \quad (5.40)$$

$$\|\nabla \psi_q^{\pm 1}\|_{L^\infty} \leq e^{C\tilde{U}} \leq C, \quad (5.41)$$

$$\|\nabla \psi_q^{\pm 1} - Id\|_{L^\infty} \leq e^{C\tilde{U}} - 1 \leq e^{C\lambda_1} - 1 \leq C\lambda_1, \quad (5.42)$$

$$\|\nabla^k \psi_q^{\pm 1}\|_{L^\infty} \leq C2^{(k-1)q}(e^{C\tilde{U}} - 1) \leq C2^{(k-1)q}\lambda_1, \text{ for } k = 2, 3, \quad (5.43)$$

where $\tilde{U}(t) = \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau \leq CX_{p_2}(t) \leq C\lambda_1$.

As for $\Delta_j \nabla \tilde{L}_q$, we can use the fact that, owing to the chain rule,

$$\Delta_j \nabla \tilde{L}_q = \Delta_j (\partial_l L_q \circ \psi_q \nabla \psi_{q,l}).$$

Then

$$\|\Delta_j \nabla \tilde{L}_q\|_{L^p} \leq C \|\nabla L_q \circ \psi_q\|_{L^p} \|\nabla \psi_q\|_{L^\infty} \leq C \|\nabla L_q\|_{L^p},$$

combining the estimate (5.34), we get

$$\|\Delta_j \nabla \tilde{L}_q\|_{L^p} \leq C \|\nabla \Delta_q L\|_{L^p} + C \|\nabla v\|_{L^\infty} \sum_{q' \sim q} \|\nabla \Delta_{q'} E\|_{L^p}.$$

Using the Vishik's trick used in [34], we can get

$$\|\Delta_j \nabla \tilde{L}_q\|_{L^p} \leq C2^{q-j} \left(\|\nabla \Delta_q L\|_{L^p} + \|\nabla v\|_{L^\infty} \sum_{q' \sim q} \|\nabla \Delta_{q'} E\|_{L^p} \right), \quad (5.44)$$

where we use the facts that

$$\|\Delta_j \nabla \tilde{L}_q\|_{L^p} \leq C2^{-j} \|\nabla^2 (L_q \circ \psi_q)\|_{L^p},$$

$$\partial_k \partial_l (L_q \circ \psi_q) = (\partial_m \partial_n L_q \circ \psi_q) \partial_k \psi_{q,m} \partial_l \psi_{q,n} + (\partial_m L_q \circ \psi_q) \partial_k \partial_l \psi_{q,m},$$

and

$$\|\nabla^2 (L_q \circ \psi_q)\|_{L^p} \leq C2^q \|\nabla L_q\|_{L^p}.$$

Similar, we get

$$\|\Delta_j \tilde{G}_q\|_{L^p} \leq C2^{q-j} \left(\|\Delta_q G\|_{L^p} + \|\nabla v\|_{L^\infty} \sum_{q' \sim q} \|\Delta_{q'} v\|_{L^p} \right). \quad (5.45)$$

Let us now turn to the study of ∇R_q^3 . As a preliminary step, we have

$$\|\Delta_j \nabla R_q^3\|_{L^p} \leq C2^{-j} \|\Delta_j \nabla^2 R_q^3\|_{L^p} \leq C2^{-j} \|\nabla^2 R_q^3\|_{L^p}.$$

Using the chain rule and Hölder inequality, we get

$$\begin{aligned} & \|\nabla^2 R_q^3\|_{L^p} \\ & \leq C \|\nabla^3 \tilde{v}_q\| \|\nabla \phi_q \circ \psi_q - Id\|_{L^\infty} + C \|\nabla^2 \tilde{v}_q\| \|\nabla^2 \phi_q \circ \psi_q\|_{L^\infty} \|\nabla \psi_q\|_{L^\infty} \\ & \quad + C \|\nabla \tilde{v}_q\| (\|\nabla^3 \phi_q \circ \psi_q\|_{L^\infty} \|\nabla \psi_q\|_{L^\infty}^2 + \|\nabla^2 \phi_q \circ \psi_q\|_{L^\infty} \|\nabla^2 \psi_q\|_{L^\infty}) \end{aligned}$$

and

$$\|\Delta_j \nabla R_q^3\|_{L^p} \leq C2^{3q-j} \lambda_1 \|v_q\|_{L^p}. \quad (5.46)$$

Similar, we have

$$\|\Delta_j R_q^1\|_{L^p} \leq C2^{q-j} \lambda_1 \|\nabla E_q\|_{L^p}, \quad (5.47)$$

$$\|\Delta_j R_q^2\|_{L^p} \leq C 2^{3q-j} \lambda_1 \|v_q\|_{L^p}. \quad (5.48)$$

From (5.38)–(5.39), (5.44)–(5.48), we have for $2^j > R_0$,

$$\begin{aligned} & 2^j \|\Delta_j \tilde{c}_q\|_{L_t^\infty(L^p) \cap L_t^1(L^p)} + \|\Delta_j \tilde{v}_q\|_{L_t^\infty(L^p)} + 2^{2j} \|\Delta_j \tilde{v}_q\|_{L_t^1(L^p)} \\ & \lesssim \|(\Delta_j \Delta_q \nabla E_0, \Delta_j \Delta_q v_0)\|_{L^p} + 2^{q-j} \|(\Delta_q \nabla L, \Delta_q G)\|_{L_t^1(L^p)} \\ & + 2^{q-j} \sum_{q' \sim q} \int_0^t \|\nabla v\|_{L^\infty} \|(\nabla E_{q'}, v_{q'})\|_{L^p} d\tau + 2^{q-j} \lambda_1 \left(\|\nabla E_q\|_{L_t^1(L^p)} + 2^{2q} \|v_q\|_{L_t^1(L^p)} \right). \end{aligned}$$

Since

$$v_q = S_{q-N_0} \tilde{v}_q \circ \phi_q + \sum_{j \geq q-N_0} \Delta_j \tilde{v}_q \circ \phi_q, \quad (5.49)$$

$$c_q = S_{q-N_0} \tilde{c}_q \circ \phi_q + \sum_{j \geq q-N_0} \Delta_j \tilde{c}_q \circ \phi_q, \quad (5.50)$$

from the following estimate ([7], Lemma 2.3),

$$\|S_j(\Delta_q f \circ \psi_q^{\pm 1})\|_{L^p} \leq C(\lambda_1 + 2^{j-q}) \|\Delta_q f\|_{L^p}, \quad (5.51)$$

we have for all $2^q > R_0 2^{N_0}$,

$$\begin{aligned} & 2^q \|c_q\|_{L_t^\infty(L^p) \cap L_t^1(L^p)} + \|v_q\|_{L_t^\infty(L^p)} + 2^{2q} \|v_q\|_{L_t^1(L^p)} \\ & \leq C 2^{3N_0} \left(\|(\Delta_q \nabla E_0, \Delta_q v_0)\|_{L^p} + \|(\Delta_q \nabla L, \Delta_q G)\|_{L_t^1(L^p)} + \sum_{q' \sim q} \int_0^t \|\nabla v\|_{L^\infty} \|(\nabla E_{q'}, v_{q'})\|_{L^p} d\tau \right) \\ & + C 2^{3N_0} \lambda_1 \left(2^q \|E_q\|_{L_t^\infty(L^p) \cap L_t^1(L^p)} + \|v_q\|_{L_t^\infty(L^p)} + 2^{2q} \|v_q\|_{L_t^1(L^p)} \right) \\ & + C 2^{-N_0} \left(2^q \|c_q\|_{L_t^\infty(L^p) \cap L_t^1(L^p)} + \|v_q\|_{L_t^\infty(L^p)} + 2^{2q} \|v_q\|_{L_t^1(L^p)} \right). \end{aligned}$$

Choosing N_0 to be an integer such that $C 2^{-N_0} \in (\frac{1}{4}, \frac{1}{2})$, we have for all $2^q > R_0 2^{N_0}$,

$$\begin{aligned} & 2^q \|c_q\|_{L_t^\infty(L^p) \cap L_t^1(L^p)} + \|v_q\|_{L_t^\infty(L^p)} + 2^{2q} \|v_q\|_{L_t^1(L^p)} \\ & \leq C \left(\|(\Delta_q \nabla E_0, \Delta_q v_0)\|_{L^p} + \|(\Delta_q \nabla L, \Delta_q G)\|_{L_t^1(L^p)} + \sum_{q' \sim q} \int_0^t \|\nabla v\|_{L^\infty} \|(\nabla E_{q'}, v_{q'})\|_{L^p} d\tau \right) \\ & + C \lambda_1 \left(2^q \|E_q\|_{L_t^\infty(L^p) \cap L_t^1(L^p)} + \|v_q\|_{L_t^\infty(L^p)} + 2^{2q} \|v_q\|_{L_t^1(L^p)} \right). \end{aligned}$$

From Lemma 5.1, we have for $R_0 2^{N_0-4} \leq 2^q \leq R_0 2^{N_0}$,

$$\|(\nabla E_q, v_q)\|_{L_t^\infty(L^p)} \leq C Y_s(t) \leq C Y_{s,0},$$

and

$$\begin{aligned} & \sum_{2^q > R_0 2^{N_0}} 2^{q(\frac{N}{p}-1)} \left(2^q \|c_q\|_{L_t^\infty(L^p) \cap L_t^1(L^p)} + \|v_q\|_{L_t^\infty(L^p)} + 2^{2q} \|v_q\|_{L_t^1(L^p)} \right) \\ & \leq C Y_{s,0} + C \sum_{2^q > R_0 2^{N_0}} 2^{\frac{N}{p}-1} \left(\|(\Delta_q \nabla E_0, \Delta_q v_0)\|_{L^p} + \|(\Delta_q \nabla L, \Delta_q G)\|_{L_t^1(L^p)} \right) \\ & + C \int_0^t \|\nabla v\|_{L^\infty} \sum_{2^q > R_0 2^{N_0}} 2^{\frac{N}{p}-1} \|(\nabla E_q, v_q)\|_{L^p} d\tau \\ & + C \lambda_1 \sum_{2^q > R_0 2^{N_0}} 2^{\frac{N}{p}-1} \left(2^q \|E_q\|_{L_t^\infty(L^p) \cap L_t^1(L^p)} + \|v_q\|_{L_t^\infty(L^p)} + 2^{2q} \|v_q\|_{L_t^1(L^p)} \right). \quad (5.52) \end{aligned}$$

From (5.10)₄ and (5.30), we have

$$\begin{aligned}
& \|E^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{N}{p}}) \cap \tilde{L}_t^1(\dot{B}_{p,1}^{\frac{N}{p}})} \\
& \leq \|c^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{N}{p}}) \cap \tilde{L}_t^1(\dot{B}_{p,1}^{\frac{N}{p}})} + C\|(E^2)^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{N}{p}}) \cap \tilde{L}_t^1(\dot{B}_{p,1}^{\frac{N}{p}})} \\
& \leq \|c^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{N}{p}}) \cap \tilde{L}_t^1(\dot{B}_{p,1}^{\frac{N}{p}})} + C\|E\|_{\tilde{L}_t^\infty(\dot{B}_{p_2,1}^{\frac{N}{p_2}}) \cap \tilde{L}_t^1(\dot{B}_{p_2,1}^{\frac{N}{p_2}})} \|E\|_{L_t^\infty(\dot{B}_{p,1}^{\frac{N}{p}})} \\
& \leq \|c^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{N}{p}}) \cap \tilde{L}_t^1(\dot{B}_{p,1}^{\frac{N}{p}})} + C\lambda_1(\|E^l\|_{\tilde{L}_t^\infty(\dot{B}_{2,r}^{s+1})} + \|E^h\|_{L_t^\infty(\dot{B}_{p,1}^{\frac{N}{p}})}).
\end{aligned}$$

When λ_1 is small enough, $C\lambda_1 < \frac{1}{2}$, we have

$$\|E^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{N}{p}}) \cap \tilde{L}_t^1(\dot{B}_{p,1}^{\frac{N}{p}})} \leq \|c^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{N}{p}}) \cap \tilde{L}_t^1(\dot{B}_{p,1}^{\frac{N}{p}})} + CY_{s,0}. \quad (5.53)$$

From (5.52)–(5.53) and

$$\sum_{R_0 < 2^q \leq R_0 2^{N_0}} 2^{q\frac{N}{p}} \|c_q\|_{L_t^\infty(L^p) \cap L_t^1(L^p)} \leq CY_s(t) \leq CY_{s,0},$$

we have

$$\begin{aligned}
& \|E^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{N}{p}}) \cap \tilde{L}_t^1(\dot{B}_{p,1}^{\frac{N}{p}})} + \|v^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{N}{p}-1}) \cap \tilde{L}_t^1(\dot{B}_{p,1}^{\frac{N}{p}+1})} \\
& \leq CY_{s,0} + CZ_{p,0} + C\|L^h\|_{\tilde{L}_t^1(\dot{B}_{p,1}^{\frac{N}{p}})} + C\|G^h\|_{\tilde{L}_t^1(\dot{B}_{p,1}^{\frac{N}{p}-1})} \\
& + C \int_0^t \|\nabla v\|_{L^\infty} \left(\|E^h\|_{\tilde{L}_\tau^\infty(\dot{B}_{p,1}^{\frac{N}{p}})} + \|v^h\|_{\tilde{L}_\tau^\infty(\dot{B}_{p,1}^{\frac{N}{p}-1})} \right) d\tau \\
& + C\lambda_1 \left(\|E^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{N}{p}}) \cap \tilde{L}_t^1(\dot{B}_{p,1}^{\frac{N}{p}})} + \|v^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{N}{p}-1}) \cap \tilde{L}_t^1(\dot{B}_{p,1}^{\frac{N}{p}+1})} \right).
\end{aligned}$$

When λ_1 small enough, $C\lambda_1 < \frac{1}{2}$, using the Gronwall's inequality, we get

$$\begin{aligned}
& \|E^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{N}{p}}) \cap \tilde{L}_t^1(\dot{B}_{p,1}^{\frac{N}{p}})} + \|v^h\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{N}{p}-1}) \cap \tilde{L}_t^1(\dot{B}_{p,1}^{\frac{N}{p}+1})} \\
& \leq CY_{s,0} + CZ_{p,0} + C\|L^h\|_{\tilde{L}_t^1(\dot{B}_{p,1}^{\frac{N}{p}})} + C\|G^h\|_{\tilde{L}_t^1(\dot{B}_{p,1}^{\frac{N}{p}-1})}.
\end{aligned} \quad (5.54)$$

Finally, from Proposition 3.1, we get

$$\begin{aligned}
\|L^h\|_{\tilde{L}_t^1(\dot{B}_{p,1}^{\frac{N}{p}})} & \leq \|\nabla v E\|_{\tilde{L}_t^1(\dot{B}_{p,1}^{\frac{N}{p}})} + \|\partial_i(T'_E v_i)\|_{\tilde{L}_t^1(\dot{B}_{p,1}^{\frac{N}{p}})} \\
& \leq C\|\nabla v\|_{\tilde{L}_t^1(\dot{B}_{p,1}^{\frac{N}{p}})} \|E\|_{\tilde{L}_t^\infty(\dot{B}_{p_2,1}^{\frac{N}{p_2}})} + C\|\nabla v\|_{\tilde{L}_t^1(\dot{B}_{p_2,1}^{\frac{N}{p_2}})} \|E\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{N}{p}})} \\
& \leq C\lambda_1 Z_p + CY_{s,0}.
\end{aligned} \quad (5.55)$$

Similarly, using Proposition 3.1, we get

$$\|T'_{\nabla v_i} \cdot v\|_{\tilde{L}_t^1(\dot{B}_{p,1}^{\frac{N}{p}-1})} = \|T'_v v\|_{\tilde{L}_t^1(\dot{B}_{p,1}^{\frac{N}{p}})} \leq C\|v\|_{\tilde{L}_t^1(\dot{B}_{p_2,1}^{\frac{N}{p_2}+1})} \|v\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{N}{p}-1})} \leq C\lambda_1 Z_p + CY_{s,0}, \quad (5.56)$$

and

$$\|E_{jk} \partial_j E_{ik}\|_{\tilde{L}_t^1(\dot{B}_{p,1}^{\frac{N}{p}-1})} \leq \|E^2\|_{\tilde{L}_t^1(\dot{B}_{p,1}^{\frac{N}{p}})} \leq C\|E\|_{\tilde{L}_t^1(\dot{B}_{p_2,1}^{\frac{N}{p_2}})} \|E\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{N}{p}})} \leq C\lambda_1 Z_p + CY_{s,0}. \quad (5.57)$$

From (1.6), we have

$$\Delta p = \partial_i(E_{jk} \partial_j E_{ik}) - \partial_i(v_j \partial_j v_i)$$

$$= \partial_i(E_{jk}\partial_j E_{ik}) - \partial_i(T'_{\partial_j v_i} v_j) - \partial_j(T_{\partial_i v_j} v_i),$$

then,

$$\begin{aligned} \|G\|_{\tilde{L}_t^1(\dot{B}_{p,1}^{\frac{N}{p}-1})} &\leq \|\nabla p\|_{\tilde{L}_t^1(\dot{B}_{p,1}^{\frac{N}{p}-1})} + \|E_{jk}\partial_j E_{ik}\|_{\tilde{L}_t^1(\dot{B}_{p,1}^{\frac{N}{p}-1})} + \|T'_{\nabla v} v\|_{\tilde{L}_t^1(\dot{B}_{p,1}^{\frac{N}{p}-1})} \\ &\leq C\lambda_1 Z_p + CY_{s,0}. \end{aligned} \quad (5.58)$$

From (5.54)–(5.55) and (5.58), we have

$$Z_p \leq CY_{s,0} + CZ_{p,0} + C\lambda_1 Z_p.$$

When $C\lambda_1 \leq \frac{1}{2}$, we can obtain (5.32). \square

From a standard bootstrap argument, we obtain the following proposition.

Proposition 5.1. *Let (v, E) be a solution to (1.1) which belongs to $V_{p_1, r}^s(T)$ with s, p_1 and r satisfying the condition of Theorem 1.2. There exist two constants λ and C depending only on N, p_2 and s such that if $X_{p_2, 0} \leq \lambda$ then*

$$X_{p_i}(t) \leq CX_{p_i, 0} \text{ for all } t \in [0, T] \text{ and } i = 1, 2.$$

6 The proof of the global existence theorem

So, let us proceed to the proof of global existence under the assumptions of Theorem 1.2. To simplify the presentation, we assume throughout that $r < \infty$. The case $r = \infty$ follows from similar arguments. It is only a matter of replacing the strong topology in $\dot{B}_{2, r}^s$ or $\dot{B}_{2, r}^{s+1}$ by weak topology.

First step: smooth solutions. We smooth out the initial data as follows,

$$v_{0, n} = S_n v_0, \quad E_{0, n} = S_n E_0.$$

Then, $(v_{0, n}, E_{0, n})_{n \in \mathbb{N}}$ satisfy

$$v_{0, n} \in \dot{B}_{2, r}^s \cap \dot{B}_{2, 1}^{\frac{N}{2}-1}, \quad E_{0, n} \in \dot{B}_{2, r}^s \cap \dot{B}_{2, 1}^{\frac{N}{2}},$$

and

$$u_{0, n} \rightarrow u_0 \text{ in } \dot{B}_{2, r}^s, \quad u_{0, n}^h \rightarrow u_0^h \text{ in } \dot{B}_{p_1, 1}^{\frac{N}{p_1}-1}, \quad (6.1)$$

$$E_{0, n}^l \rightarrow E_0^l \text{ in } \dot{B}_{2, r}^s, \quad E_{0, n}^h \rightarrow E_0^h \text{ in } \dot{B}_{2, r}^{s+1} \cap \dot{B}_{p_1, 1}^{\frac{N}{p_1}}. \quad (6.2)$$

From Theorem 1.1, for all $n \in \mathbb{N}$, we obtain a maximal solution (v_n, E_n) over the time interval $[0, T_n^*)$ such that for all $T < T_n^*$, we have

$$v_n \in \left(\tilde{C}_T(\dot{B}_{2, 1}^{\frac{N}{2}-1}) \cap L_T^1(\dot{B}_{2, 1}^{\frac{N}{2}+1}) \right)^N \text{ and } E_n \in \left(\tilde{C}_T(\dot{B}_{2, 1}^{\frac{N}{2}}) \right)^{N \times N}.$$

We claim that (v_n, E_n) is in $V_{p_1, r}^s(T)$ (and thus also in $V_{p_2, r}^s(T)$) for all $T < T_n^*$. Because $p_1 \geq 2$, we have

$$v_n^h \in \left(\tilde{C}_T(\dot{B}_{p_1, 1}^{\frac{N}{p_1}-1}) \cap L_T^1(\dot{B}_{p_1, 1}^{\frac{N}{p_1}+1}) \right)^N \text{ and } E_n^h \in \left(\tilde{C}_T(\dot{B}_{p_1, 1}^{\frac{N}{p_1}}) \cap \tilde{C}_T(\dot{B}_{2, r}^s) \right)^{N \times N}.$$

Then, from the equation, we can show that the low frequencies of E_n and of v_n are on $\tilde{C}_T(\dot{B}_{2, r}^s)$ for all $T < T_n^*$.

Second step: global existence and uniform bounds for (v_n, E_n) . From Proposition 5.1, there exists a constant λ such that if $X_{p_2, 0}^n \leq \lambda$, then

$$X_{p_i}^n(t) \leq C_1 X_{p_i, 0}^n, \text{ for all } i = 1, 2 \text{ } t \in [0, T]. \quad (6.3)$$

If we assume that $X_{p_2, 0} \leq \frac{\lambda}{2}$, then properties (6.1)–(6.2) guarantees that the above smallness condition for the smoothed out initial data are satisfied for all large enough n .

Assume, by contradiction, that T_n^* is finite. Then applying Proposition 5.1, we get

$$X_2^n(t) \leq CX_{2,0}^n \text{ for all } t \in [0, T_n^*).$$

So, $v_n \in (\tilde{L}_{T_n^*}^\infty(\dot{B}_{2,1}^{\frac{N}{2}-1}))^N$ and $E_n \in (\tilde{L}_{T_n^*}^\infty(\dot{B}_{2,1}^{\frac{N}{2}}))^{N \times N}$. Then, one can easily obtain that (v_n, E_n) may be continued beyond T_n^* into a solution $(\tilde{v}_n, \tilde{E}_n)$ of (1.6) which coincides with (v_n, E_n) on $[0, T_n^*)$ and such that, for some $T > T_n^*$,

$$v_n \in \left(\tilde{C}_T(\dot{B}_{2,1}^{\frac{N}{2}-1}) \cap L_T^1(\dot{B}_{2,1}^{\frac{N}{2}+1}) \right)^N \text{ and } E_n \in \left(\tilde{C}_T(\dot{B}_{2,1}^{\frac{N}{2}}) \right)^{N \times N}.$$

And one can easily obtain that $(\tilde{v}_n, \tilde{E}_n) \in V_{p_1, r}^s(T)$. This stands in contradiction with the definition of T_n^* . Thus, $T_n^* = \infty$ and (6.3) holds true globally.

Third step: passing to the limit. Using the Aubin-Lions type argument [6], we easily have that, up to extraction, (v_n, E_n) converges to (v, E) , and $(v, E) \in V_{p, r}^s(\infty)$, $p = p_1$ or p_2 , (v, E) is the solution of (1.6).

Then, we can easily obtain that $(v, E) \in \mathcal{E}_T^{\frac{N}{p_1}}$. From the following local result in [30], we can obtain the uniqueness of the solution when $p_1 \leq 2N$. This finishes the proof of Theorem 1.2.

Theorem 6.1 ([30]). *Suppose that the initial data satisfies the incompressible constraints (1.3), $v_0 \in (\dot{B}_{p,1}^{\frac{N}{p}-1})^N$ and $E_0 \in (\dot{B}_{p,1}^{\frac{N}{p}})^{N \times N}$, $p \in [1, 2N]$. Then there exist $T > 0$ and a unique local solution for system (1.6) that satisfies $(v, E) \in \mathcal{E}_T^{\frac{N}{p}}$, where $\mathcal{E}_T^s = \left(L^1([0, T]; \dot{B}_{p,1}^{s+1}) \cap C([0, T]; \dot{B}_{p,1}^{s-1}) \right)^N \times \left(C([0, T]; \dot{B}_{p,1}^s) \right)^{N \times N}$.*

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